THREE-STAGE SAMPLING WITH VARYING PROBABILITIES OF SELECTION

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1. Summary

The paper deals with three-stage sampling with varying probabilities in every stage of selection. A system of selections without replacement is also discussed, which consists in reducing the chances of reselection of a unit after every selection for a given quantity ϵ. This is done at every stage. The system implies selection with replacement for $\epsilon = 0$, and can also give an approximation for selection with complete non-replacement, in which case the chance of the reselection of a once selected unit drops immediately to zero.

In Chapter 2, well-known formulæ for a three-stage sampling with equal probabilities of selection are quoted, introducing symbols which will be used later. The formulæ are generalized in Chapter 3 to the case mentioned above. The results are formulæ (20) to (27) which apply to the unbiassed estimate of the universe total for a given system of probabilities of selection. Formulæ are given for the variance error, estimated variance error, estimated universe parameters occurring in the variance-error formula, expectations of sample parameters occurring in the formula for the estimated variance error and finally for the estimated variance error for a larger or smaller sample where the estimate is built up on the data of the actually selected sample. Formulæ (31) to (42) are general results for any linear estimator providing biassed estimates of the total or mean. Three cases are worked out to show how a linear estimator can be adjusted to give an unbiassed estimate.

Chapter 4 contains proofs for the formulæ given in Chapter 3. They are derived as special cases of formulæ for a selection with equal probabilities, making an extension of the universe and a transformation of the values of the observed variate.

In Chapter 5, the necessary modifications are indicated for the application of the results of the preceding chapter to sampling designs which represent a simplification of three-stage sampling (*e.g.*, stratified two-stage sampling, two-stage cluster sampling, etc.). After this,

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India we are doing a certain amount of useful work on the study of crop pests, but a good deal more requires to be done.

I have spoken at some length on the urgent need of more work in what I have called the second area of the application of statistical methods in agriculture which is based on the use of the design of experiments. I venture to think that is a very special responsibility of the Indian Council of Agricultural Research and your Society. I have tried to indicate the vast areas of ignorance which require urgent attention. These are subjects of basic importance for crop production and crop protection not only for the immediate future but for purposes of agricultural planning over a period of 15, 20 or 30 years or more. I have referred to five important topics, namely, the need of fertilizers, water requirements of crops, improved methods of cultivation, improved varieties, and methods of crop protection. Work in this field offers almost unlimited scope for scientific research using the basic methods of controlled experiments in which the theory of design of experiments would be a most important tool. This is a vast field which, in comparison with statistical surveys for purposes of estimation in agriculture, does not seem to have been fully cultivated in India. This is a thought which I should like to commend to you. I should also like to draw your attention to the need of hard and systematic work to build up the scientific foundations for crop planning for the next 30 years to change the whole pattern of living in India.

special cases are dealt with in which the varying probabilities are proportionate to size (size meaning the number of tertiary units in primary or secondary sampling units), and later in which the selection is done with replacement or with complete non-replacement. Finally the application of estimators which provide a biassed estimate is discussed, and it is stressed that the estimated variance error of such estimators does not depend upon the probabilities actually used in the selection of the sample, provided that the sum of chances of selection for each stage is kept constant. The results of this paper make obvious a generalization of the obtained formulæ to a bigger number of sampling stages.

2. Introduction and Basic Formulæ

This paper concerns three-stage sampling with varying probabilities. It is taken as granted that the results in this chapter for three-stage sampling with equal probabilities of selection in each stage are known and proved. Let us have a universe with primary sampling units denoted by the letter i ($i = 1, 2, K$) having in the i-th primary unit some secondary units denoted by j ($j = 1, 2, \cdots, L_i$), and within the j-th secondary unit of the i-th primary unit let us have tertiary sampling units denoted by h ($h = 1, 2, \cdots, N_{ij}$). The character observed should be denoted by ξ_{ijh}, the subscripts referring to the serial number of the primary, secondary and tertiary units respectively. From this universe a three-stage sample is selected with equal probabilities, and without replacement, including in it k primary units, l_i secondary units out of the selected i-th primary unit, and n_{ij} tertiary units out of the selected i-th primary and j-th secondary unit. Our aim is to determine from the sample an unbiassed estimate and to estimate the variance error of the universe total:

$$\Xi = \sum_1 \sum_2 \sum_3 \xi_{ijh} \tag{1}$$

or the universe mean per tertiary unit:

$$\bar{\xi}_{ijh} = \frac{\sum_1 \sum_2 \sum_3 \xi_{ijh}}{\sum_1 \sum_2 N_{ij}} \tag{2}$$

where Σ means complete summation, the subscript referring to the stage of the sampling units.

It is known that an unbiassed estimate of (1) is obtained by the estimator

$$x = \frac{K}{k} \underset{1}{S} \frac{L_i}{l_i} \underset{2}{S} \frac{N_{ij}}{n_{ij}} \underset{3}{S} x_{ijh}$$

where S means summation only over sample units, the subscript referring to the stage of sampling. As the indexes i, j, h refer only to selected values from the universe, because the sample is smaller, we shall indicate the value of the observed variate in the sample by x_{ijh} (instead of ξ_{ijh}), where the subscripts refer to the serial numbers of the sampling units in the sample. Also, if we write $\sum_1 \sum_2 N_{ij} = N$, then an unbiassed estimate of (2) is obtained by the estimator:

$$\bar{x} = \frac{X}{N} = \frac{1}{N} \cdot \frac{K}{k} \underset{1}{S} \frac{L_i}{l_i} \underset{2}{S} \frac{N_{ij}}{n_{ij}} \underset{3}{S} x_{ijh} \tag{4}$$

This statement will be denoted symbolically by:

$$EX = \Xi, \; E\bar{x} = \bar{\xi} \tag{5}$$

or by

$$\exists \Xi = X, \; \exists \bar{\xi} = \bar{x} \tag{6}$$

where E means "mathematical expectation", whilst $\exists$ means "unbiassed estimate".

Further on the variance error of (3) is given by[1]:

$$\begin{aligned} \text{Var}(X) = \left(\frac{K}{k}\right)^2 \cdot \Bigg\{ & k \cdot \frac{K-k}{K-1} \sigma_1^2 \left[\sum_2 \sum_3 \xi_{ijh}\right] \\ & + \frac{k}{K} \sum_1 \frac{L_i - l_i}{L_i - 1} \left(\frac{L_i}{l_i}\right)^2 \cdot l_i \sigma_2^2 \left[\sum_3 \xi_{ijh}\right] \\ & + \frac{k}{K} \sum_1 \left(\frac{L_i}{l_i}\right)^2 \cdot l_i \cdot \frac{1}{L_i} \sum_2 \frac{N_{ij} - n_{ij}}{N_{ij} - 1} \cdot \left(\frac{N_{ij}}{n_{ij}}\right)^2 \\ & \times n_{ij} \sigma_3^2 [\xi_{ijh}] \Bigg\} \end{aligned} \tag{7}$$

Here the symbol "Var" denotes the variance error, *i.e.*,

$$\text{Var}(X) = E[X - E(X)]^2 \tag{8}$$

The symbols σ^2 in (7) are parameters of the universe, and their general meaning is

$$\sigma_r^2 [t] = \frac{\sum_r \left(t - \frac{\sum_r t}{\sum_r 1} \right)^2}{\sum_r 1} \tag{9}$$

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[1] Compare Ref. [3] and in a simplified form in Ref. [4], formula (35).

where r refers to the sampling stage. Thus the formula (9) applied to the parameters mentioned in (7) gives:

$$\sigma_1^2\left[\sum_2\sum_3 \xi_{ijh}\right] = \frac{1}{K}\sum_1\left(\sum_2\sum_3 \xi_{ijh} - \frac{1}{K}\sum_1\sum_2\sum_3 \xi_{ijh}\right)^2$$

$$\sigma_2^2\left[\sum_3 \xi_{ijh}\right] = \frac{1}{L_i}\sum_2\left(\sum_3 \xi_{ijh} - \frac{1}{L_i}\sum_2\sum_3 \xi_{ijh}\right)^2 \tag{10}$$

$$\sigma_3^2\,[\xi_{ijh}] = \frac{1}{N_{ij}}\sum_3\left(\xi_{ijh} - \frac{1}{N_{ij}}\sum_3 \xi_{ijh}\right)^2$$

As for the variance error of (4), we have:

$$\text{Var}\,(\bar{x}) = \frac{1}{N^2}\,\text{Var}\,(X) \tag{11}$$

and we have to use in this formula the expression (7).

Further we know that an unbiassed estimate of (7) is given by the estimator[2]:

$$\begin{aligned} \mathcal{E}\,\text{Var}\,(X) = \left(\frac{K}{k}\right)^2 \cdot \Bigg\{ & k\left(1-\frac{k}{K}\right) s_1^2\left[\frac{L_i}{l_i}\underset{2}{S}\frac{N_{ij}}{n_{ij}}\underset{3}{S}x_{ijh}\right] \\ & + \frac{k}{K}\underset{1}{S}\left(1-\frac{l_i}{L_i}\right)\left(\frac{L_i}{l_i}\right)^2 \cdot l_i s_2^2\left[\frac{N_{ij}}{n_{ij}}\underset{3}{S}x_{ijh}\right] \\ & + \frac{k}{K}\underset{1}{S}\left(\frac{L_i}{l_i}\right)^2 \cdot l_i \cdot \frac{1}{L_i}\underset{2}{S}\left(1-\frac{n_{ij}}{N_{ij}}\right)\cdot\left(\frac{N_{ij}}{n_{ij}}\right)^2 \\ & \times n_{ij}s_3^2\,[x_{ijh}]\Bigg\} \end{aligned} \tag{12}$$

The symbols s^2 in (12) are sample parameters, and their general meaning is

$$s_r^2\,[t] = \frac{\underset{r}{S}\left(t - \dfrac{\underset{r}{S}t}{\underset{r}{S}1}\right)^2}{\underset{r}{(S1)} - 1} \tag{13}$$

[2] See Ref. [3] although the formula is given there only for a two-stage sample; also see Ref. [1], Chapter 10, formula (37).

where r refers to the sampling stage. Thus the formula (13) applied to the parameters mentioned in (12) gives:

$$s_1^2\left[\frac{L_i}{l_i}\underset{2}{S}\frac{N_{ij}}{n_{ij}}\underset{3}{S}x_{ijh}\right]$$

$$=\frac{1}{k-1}\underset{1}{S}\left(\frac{L_i}{l_i}\underset{2}{S}\frac{N_{ij}}{n_{ij}}\underset{3}{S}x_{ijh}-\frac{1}{k}\underset{1}{S}\frac{L_i}{l_i}\underset{2}{S}\frac{N_{ij}}{n_{ij}}\underset{3}{S}x_{ijh}\right)^2$$

$$s_2^2\left[\frac{N_{ij}}{n_{ij}}\underset{3}{S}x_{ijh}\right]$$

$$=\frac{1}{l_i-1}\underset{2}{S}\left(\frac{N_{ij}}{n_{ij}}\underset{3}{S}x_{ijh}-\frac{1}{l_i}\underset{2}{S}\frac{N_{ij}}{n_{ij}}\underset{3}{S}x_{ijh}\right)^2$$

$$s_3^2\,[x_{ijh}]$$

$$=\frac{1}{n_{ij}-1}\underset{3}{S}\left(x_{ijh}-\frac{1}{n_{ij}}\underset{3}{S}x_{ijh}\right)^2 \tag{14}$$

As for the estimated variance error of (11), we have

$$\mathcal{E}\,\mathrm{Var}\,(\bar{x})=\frac{1}{N^2}\cdot\mathcal{E}\,\mathrm{Var}\,(X) \tag{15}$$

and we have to use in this formula the expression (12).

Instead of using formula (12), the parameters of the universe, σ^2, in (7), can be estimated from the sample by expressions which follow; by inserting these expressions in (7), we also obtain an unbiassed estimate of Var (X):[3]

$$\underset{123}{\mathcal{E}}\,\sigma_1^2\left[\underset{2}{\Sigma}\underset{3}{\Sigma}\,\xi_{ijh}\right]$$

$$=\frac{K-1}{K}\left\{s_1^2\left[\frac{L_i}{l_i}\underset{2}{S}\frac{N_{ij}}{n_{ij}}\underset{3}{S}x_{ijh}\right]\right.$$

$$-\frac{1}{k}\underset{1}{S}\left(1-\frac{l_i}{L_i}\right)\cdot\left(\frac{L_i}{l_i}\right)^2\cdot l_i s_2^2\left[\frac{N_{ij}}{n_{ij}}\underset{3}{S}x_{ijh}\right]$$

$$-\frac{1}{k}\underset{1}{S}\left(\frac{L_i}{l_i}\right)^2\cdot l_i\cdot\frac{1}{L_i}\underset{2}{S}\left(1-\frac{n_{ij}}{N_{ij}}\right)\cdot\left(\frac{N_{ij}}{n_{ij}}\right)^2$$

$$\left.\times\, n_{ij}s_3^2\,[x_{ijh}]\right\}$$

[3] See Ref. [1], Chapter 10, formula (34), for a two-stage sample.

$$\underset{23}{\exists}\,\sigma_2^2\left[\sum_3 \xi_{ijh}\right]$$

$$= \frac{L_i-1}{L_i}\left\{s_2^2\left[\frac{N_{ij}}{n_{ij}}\underset{3}{S}x_{ijh}\right] - \frac{1}{l_i}\underset{2}{S}\left(1-\frac{n_{ij}}{N_{ij}}\right)\right.$$

$$\left.\times\left(\frac{N_{ij}}{n_{ij}}\right)^2\cdot n_{ij}s_3^2\,[x_{ijh}]\right\}$$

$$\underset{3}{\exists}\,\sigma_3^2\,[\xi_{ijh}]$$

$$= \frac{N_{ij}-1}{N_{ij}}\,s_3^2\,[x_{ijh}] \qquad (16)$$

Here $\underset{123}{\exists}$ means an unbiassed estimate over all three stages, and this symbol is equivalent to the symbol $\exists$; $\underset{23}{\exists}$ means an unbiassed estimate over the second and third stage, $\underset{3}{\exists}$ only over the third stage; if we introduce the following symbolical operations:

$$E\,\exists t = t;\quad \underset{123}{E}\,\underset{123}{\exists}t = t;\quad E = \underset{123}{E} = \underset{1}{E}\,\underset{23}{E} = \underset{1}{E}\,\underset{2}{E}\,\underset{3}{E};\quad \exists = \underset{123}{\exists} = \underset{1}{\exists}\,\underset{23}{\exists}$$

$$= \underset{1}{\exists}\,\underset{2}{\exists}\,\underset{3}{\exists};\quad \underset{23}{E}\,\exists t = \underset{1}{E}\,\underset{23}{E}\,\underset{23}{\exists}t = \underset{1}{E}t;\quad \underset{3}{E}\,\exists t = \underset{123}{E}\,\underset{3}{\exists}t = \underset{1}{E}\,\underset{2}{E}t \qquad (17)$$

then the application of the symbolical operation $\exists\left(=\underset{123}{\exists}\right)$ to both sides of (7) gives:

$$\exists\,\mathrm{Var}\,(X) = \left(\frac{K}{k}\right)^2\cdot\left\{k\cdot\frac{K-k}{K-1}\cdot\underset{123}{\exists}\,\sigma_1^2\right.$$

$$+\frac{k}{K}\cdot\underset{1}{\exists}\sum_1\frac{L_i-l_i}{L_i-1}\cdot\left(\frac{L_i}{l_i}\right)^2\cdot l_i\cdot\underset{23}{\exists}\,\sigma_2^2$$

$$+\frac{k}{K}\cdot\underset{1}{\exists}\sum_1\left(\frac{L_i}{l_i}\right)^2\cdot l_i\cdot\frac{1}{L_i}\cdot\underset{2}{\exists}\sum_2\frac{N_{ij}-n_{ij}}{N_{ij}-1}\cdot\left(\frac{N_{ij}}{n_{ij}}\right)$$

$$\left.\times\, n_{ij}\cdot\underset{3}{\exists}\,\sigma_3^2\right\}$$

With sampling with equal probabilities, we have

$$\underset{1}{E}\frac{1}{k}\underset{1}{S}t = \frac{1}{K}\sum_1 t;\quad \underset{2}{E}\frac{1}{l_i}\underset{2}{S}t = \frac{1}{L_i}\sum_2 t;\quad \underset{3}{E}\frac{1}{n_{ij}}\underset{3}{S}t = \frac{1}{N_{ij}}\sum_3 t.$$

and thus:

$$\underset{1}{\mathcal{E}} \underset{1}{\Sigma} t = \frac{K}{k} \underset{1}{S} t; \quad \underset{2}{\mathcal{E}} \underset{2}{\Sigma} t = \frac{L_i}{l_i} \underset{2}{S} t; \quad \underset{3}{\mathcal{E}} \underset{3}{\Sigma} t = \frac{N_{ij}}{n_{ij}} \underset{3}{S} t;$$

and then applying (16), we get (12).

It can be easily shown that (16) is equivalent to[4]:

$$\underset{123}{E}\, s_1^2 \left[\frac{L_i}{l_i} \underset{2}{S} \frac{N_{ij}}{n_{ij}} \underset{3}{S} x_{ijh}\right]$$

$$= \frac{K}{K-1} \sigma_1^2 \left[\underset{2}{\Sigma} \underset{3}{\Sigma} \xi_{ijh}\right] + \frac{1}{K} \underset{1}{\Sigma} \frac{L_i - l_i}{L_i - 1} \cdot \left(\frac{L_i}{l_i}\right)^2$$

$$\times\, l_i \sigma_2^2 \left[\underset{3}{\Sigma} \xi_{ijh}\right] + \frac{1}{K} \underset{1}{\Sigma} \left(\frac{L_i}{l_i}\right)^2 \cdot l_i \cdot \frac{1}{L_i} \underset{2}{\Sigma} \frac{N_{ij} - n_{ij}}{N_{ij} - 1}$$

$$\times \left(\frac{N_{ij}}{n_{ij}}\right)^2 \cdot n_{ij} \sigma_3^2 \left[\xi_{ijh}\right]$$

$$\underset{23}{E}\, s_2^2 \left[\frac{N_{ij}}{n_{ij}} \underset{3}{S} x_{ijh}\right]$$

$$= \frac{L_i}{L_i - 1} \sigma_2^2 \left[\underset{3}{\Sigma} \xi_{ijh}\right] + \frac{1}{L_i} \underset{2}{\Sigma} \frac{N_{ij} - n_{ij}}{N_{ij} - 1} \cdot \left(\frac{N_{ij}}{n_{ij}}\right)^2$$

$$\times\, n_{ij} \sigma_3^2 \left[\xi_{ijh}\right]$$

$$\underset{3}{E}\, s_3^2 \left[x_{ijh}\right]$$

$$= \frac{N_{ij}}{N_{ij} - 1} \sigma_3^2 \left[\xi_{ijh}\right] \qquad (18)$$

If we apply the symbolical operation $E\left(= \underset{123}{E}\right)$ to formula (12), we obtain,

$$E \cdot \mathcal{E}\, \mathrm{Var}(X) = \mathrm{Var}(X)$$

$$= \left(\frac{K}{k}\right)^2 \cdot \left\{ k \left(1 - \frac{k}{K}\right) \cdot \underset{123}{E}\, s_1^2 \right.$$

$$+ \frac{k}{K} \cdot \underset{1}{E} \underset{1}{S} \left(1 - \frac{l_i}{L_i}\right) \cdot \left(\frac{L_i}{l_i}\right)^2 \cdot l_i \cdot \underset{23}{E} s_2^2$$

$$+ \frac{k}{K} \cdot \underset{1}{E} \underset{1}{S} \left(\frac{L_i}{l_i}\right)^2 \cdot l_i \cdot \frac{1}{L_i} \cdot \underset{2}{E} \underset{2}{S} \left(1 - \frac{n_{ij}}{N_{ij}}\right)$$

$$\left. \times \left(\frac{N_{ij}}{n_{ij}}\right)^2 \cdot n_{ij} \cdot \underset{3}{E} s_3^2 \right\}$$

[4] See also Ref. [1], Chapter 10, formula (33) for a two-stage sample; and also in a simplified form in Ref. [4], formulæ (40).

and having in mind that

$$\underset{1}{E}\underset{1}{S}t = \frac{k}{K}\underset{1}{\Sigma}t; \ \underset{2}{E}\underset{2}{S}t = \frac{l_i}{L_i}\underset{2}{\Sigma}t; \ \underset{3}{E}\underset{3}{S}t = \frac{n_{ij}}{N_{ij}}\underset{3}{\Sigma}t,$$

and using (18) we come back to (7).

Finally, if from the universe a larger or smaller sample is drawn (also without replacement), having k', l_i', n_{ij}' units instead of k, l_i, n_{ij}, we have to rewrite formula (7) concerning the variance error by using this new sample size. So, we get a variance error Var′ (X). However, if we want to estimate this variance error from a former sample with k, l_i, n_{ij} units, we have to use in the changed formula (7) the estimates of σ_r^2 obtained from the former sample by using strictly the unchanged formulæ (16). It is easy to derive, that then we have:

$$\begin{aligned} \mathcal{E}\,\mathrm{Var}'(X) &= \mathcal{E}\,\mathrm{Var}(X) \\ &+ K^2\left(\frac{1}{k'} - \frac{1}{k}\right) s_1^2 \left[\frac{L_i}{l_i} \underset{2}{S} \frac{N_{ij}}{n_{ij}} \underset{3}{S} x_{ijh}\right] \\ &+ \frac{K^2}{kk'} \underset{1}{S} L_i^2 \left(\frac{1}{l_i'} - \frac{1}{l_i}\right) s_2^2 \left[\frac{N_{ij}}{n_{ij}} \underset{3}{S} x_{ijh}\right] \\ &+ \frac{K^2}{kk'} \underset{1}{S} \frac{L_i^2}{l_i' l_i} \underset{2}{S} N_{ij}^2 \left(\frac{1}{n_{ij}'} - \frac{1}{n_{ij}}\right) s_3^2 \left[x_{ijh}\right] \qquad (19) \end{aligned}$$

In our further consideration we will have in mind a more general definition of the selection without replacement. Suppose, when selecting a certain primary unit, instead of eliminating only this unit from the universe before the next draw, we eliminate from the universe ϵ such units with the same composition. To be able to do so, we must suppose that there are at least $k\epsilon$ primary units of every composition. In the same way in selecting a certain secondary or tertiary unit, we eliminate from the selected units which contain them before the next draw ϵ_i and ϵ_{ij} respectively such units with the same composition. It is also here assumed that this is possible. Such sampling is equivalent to a common sampling without replacement from a smaller universe, in which for every ϵ primary units (ϵ_i sec. units and ϵ_{ij} tertiary units) stands only one primary unit (secondary, tertiary unit). It is obvious that if the former number of sampling units K, L_i, N_{ij} is reduced to K/ϵ, L_i/ϵ_i, N_{ij}/ϵ_{ij}, then such selection, characterized with constants ϵ, ϵ_i, ϵ_{ij} from the former universe, is equivalent to the common selection without replacement in the reduced universe. This means that for such a reduced universe, all formulæ mentioned above can be applied, covering however the selection from the former universe with characteristic constants ϵ, ϵ_i, ϵ_{ij}. We shall not write these formulæ here, because they will be later given for the more general case.

3. Results for Three-Stage Samples Selected with Varying Probabilities

We consider a selection without replacement with constants ϵ, ϵ_i, ϵ_{ij}, and with varying probabilities which are used in the following way. To every primary unit in the universe a chance of selection κ_i is attached with $\sum\limits_1 \kappa_i = \kappa$, making the probability of selection of this unit in the first draw $\pi_i = \kappa_i/\kappa$. In a similar way, the j-th secondary unit in the i-th primary unit will have the chance $\lambda_{ij}\left(\sum\limits_2 \lambda_{ij} = \lambda_i\right)$, and conditional probability $\pi_{ij} = \lambda_{ij}/\lambda_i$, and the h-th tertiary unit in the i-th primary, j-th secondary unit has the chance $\nu_{ijh}\left(\sum\limits_3 \nu_{ijh} = \nu_{ij}\right)$ and conditional probability $\pi_{ijh} = \nu_{ijh}/\nu_{ij}$. For the g-th selections of the same unit, following the first one, the chances are diminished by $g\epsilon$, $g\epsilon_i$, $g\epsilon_{ij}$ for primary, secondary and tertiary units respectively. Further let us introduce the abbreviations

$$\sum_2 N_{ij} = N_i, \; \sum_1 \sum_2 N_{ij} = N; \; \frac{N_i}{N} = p_i, \; \frac{N_{ij}}{N_i} = p_{ij}, \; \frac{1}{N_{ij}} = p_{ijh}.$$

Thus $Np_i p_{ij} p_{ijh} = 1$.

Under these conditions it can be shown that the unbiassed estimate of (1) is obtained by the estimator:

$$\chi = \frac{N}{k} \underset{1}{S} \frac{p_i}{\pi_i l_i} \underset{2}{S} \frac{p_{ij}}{\pi_{ij} n_{ij}} \underset{3}{S} \frac{p_{ijh} x_{ijh}}{\pi_{ijh}}$$

$$= \frac{1}{k} \underset{1}{S} \frac{1}{\pi_i l_i} \underset{2}{S} \frac{1}{\pi_{ij} n_{ij}} \underset{3}{S} \frac{x_{ijh}}{\pi_{ijh}} \tag{20}$$

i.e.,

$$E\chi = \Xi \tag{21}$$

whilst

$$\bar{x} = \frac{\chi}{N}$$

(22) with

$$E\bar{x} = \bar{\xi} \tag{22}$$

Furthermore we have for the variance error of the estimate of the universe total:

$$\text{Var}(X) = N^2 \left\{ \frac{1}{k} \frac{\kappa - \epsilon k}{\kappa - \epsilon} \sigma_1^2 \left[\frac{p_i}{\pi_i} \sum_2 p_{ij} \sum_3 p_{ijh} \xi_{ijh} \right] + \frac{1}{k} \sum_1 \pi_i \frac{\lambda_i - \epsilon_i l_i}{\lambda_i - \epsilon_i} \cdot \frac{1}{l_i} \sigma_2^2 \left[\frac{p_i}{\pi_i} \cdot \frac{p_{ij}}{\pi_{ij}} \sum_3 p_{ijh} \xi_{ijh} \right] + \frac{1}{k} \sum_1 \pi_i \frac{1}{l_i} \sum_2 \pi_{ij} \frac{\nu_{ij} - \epsilon_{ij} n_{ij}}{\nu_{ij} - \epsilon_{ij}} \cdot \frac{1}{n_{ij}} \times \sigma_3^2 \left[\frac{p_i}{\pi_i} \cdot \frac{p_{ij}}{\pi_{ij}} \cdot \frac{p_{ijh}}{\pi_{ijh}} \xi_{ijh} \right] \right\} \quad (23)$$

in which

$$\sigma_1^2\,[t] = \sum_1 \pi_i \left(t - \sum_1 \pi_i t\right)^2$$

$$\sigma_2^2\,[t] = \sum_2 \pi_{ij} \left(t - \sum_2 \pi_{ij} t\right)^2$$

$$\sigma_3^2\,[t] = \sum_3 \pi_{ijh} \left(t - \sum_3 \pi_{ijh} t\right)^2 \quad (23')$$

The unbiassed estimate of this variance error is given by the estimator:

$$\mathcal{E}\,\text{Var}(X) = N^2 \left\{ \frac{1}{k}\left(1 - \frac{\epsilon k}{\kappa}\right) s_1^2 \left[\frac{p_i}{\pi_i l_i} \underset{2}{S} \frac{p_{ij}}{\pi_{ij} n_{ij}} \underset{3}{S} \frac{p_{ijh} x_{ijh}}{\pi_{ijh}} \right] + \frac{\epsilon}{k\kappa} \underset{1}{S} \frac{1}{l_i} \left(1 - \frac{\epsilon_i l_i}{\lambda_i}\right) s_2^2 \left[\frac{p_i}{\pi_i} \frac{p_{ij}}{\pi_{ij} n_{ij}} \underset{3}{S} \frac{p_{ijh} x_{ijh}}{\pi_{ijh}} \right] + \frac{\epsilon}{k\kappa} \underset{1}{S} \frac{\epsilon_i}{l_i \lambda_i} \underset{2}{S} \frac{1}{n_{ij}} \left(1 - \frac{\epsilon_{ij} n_{ij}}{\nu_{ij}}\right) s_3^2 \times \left[\frac{p_i}{\pi_i} \cdot \frac{p_{ij}}{\pi_{ij}} \cdot \frac{p_{ijh}}{\pi_{ijh}} x_{ijh} \right] \right\} \quad (24)$$

in which

$$s_1^2\,[t] = \frac{1}{k-1} \underset{1}{S} \left(t - \frac{1}{k} \underset{1}{S} t\right)^2$$

$$s_2^2\,[t] = \frac{1}{l_i - 1} \underset{2}{S} \left(t - \frac{1}{l_i} \underset{2}{S} t\right)^2$$

$$s_3^2\,[t] = \frac{1}{n_{ij} - 1} \underset{3}{S} \left(t - \frac{1}{n_{ij}} \underset{3}{S} t\right)^2 \quad (24\,)$$

Note, that for $\epsilon = 0$, *i.e.*, selection in the first stage *with replacement*, the estimate $\mathcal{E}$ Var (X) consists of the first term only and does not depend on ϵ_i, ϵ_{ij} at all.

The parameters of the universe in (23) are estimated by the following estimators:

$$\underset{123}{\mathcal{E}}\ \sigma_1^2\left[\frac{p_i}{\pi_i}\sum_2 p_{ij}\sum_3 p_{ijh}\xi_{ijh}\right]$$

$$= \frac{\kappa-\epsilon}{\kappa}\left\{s_1^2\left[\frac{p_i}{\pi_i l_i}\underset{2}{S}\frac{p_{ij}}{\pi_{ij}n_{ij}}\underset{3}{S}\frac{p_{ijh}x_{ijh}}{\pi_{ijh}}\right]\right.$$

$$-\frac{1}{k}\underset{1}{S}\frac{1}{l_i}\left(1-\frac{\epsilon_i l_i}{\lambda_i}\right)s_2^2\left[\frac{p_i}{\pi_i}\cdot\frac{p_{ij}}{\pi_{ij}n_{ij}}\underset{3}{S}\frac{p_{ijh}x_{ijh}}{\pi_{ijh}}\right]$$

$$\left.-\frac{1}{k}\underset{1}{S}\frac{\epsilon_i}{l_i\lambda_i}\underset{2}{S}\frac{1}{n_{ij}}\left(1-\frac{\epsilon_{ij}n_{ij}}{\nu_{ij}}\right)s_3^2\left[\frac{p_i}{\pi_i}\cdot\frac{p_{ij}}{\pi_{ij}}\frac{p_{ijh}}{\pi_{ijh}}x_{ijh}\right]\right\}$$

$$\underset{23}{\mathcal{E}}\ \sigma_2^2\left[\frac{p_i}{\pi_i}\cdot\frac{p_{ij}}{\pi_{ij}}\sum_3 p_{ijh}\xi_{ijh}\right]$$

$$= \frac{\lambda_i-\epsilon_i}{\lambda_i}\left\{s_2^2\left[\frac{p_i}{\pi_i}\cdot\frac{p_{ij}}{\pi_{ij}n_{ij}}\underset{3}{S}\frac{p_{ijh}x_{ijh}}{\pi_{ijh}}\right]\right.$$

$$\left.-\frac{1}{l_i}\underset{2}{S}\frac{1}{n_{ij}}\left(1-\frac{\epsilon_{ij}n_{ij}}{\nu_{ij}}\right)s_3^2\left[\frac{p_i}{\pi_i}\cdot\frac{p_{ij}}{\pi_{ij}}\cdot\frac{p_{ijh}}{\pi_{ijh}}x_{ijh}\right]\right\}$$

$$\underset{3}{\mathcal{E}}\ \sigma_3^2\left[\frac{p_i}{\pi_i}\cdot\frac{p_{ij}}{\pi_{ij}}\cdot\frac{p_{ijh}}{\pi_{ijh}}\xi_{ijh}\right]$$

$$= \frac{\nu_{ij}-\epsilon_{ij}}{\nu_{ij}}s_3^2\left[\frac{p_i}{\pi_i}\cdot\frac{p_{ij}}{\pi_{ij}}\cdot\frac{p_{ijh}}{\pi_{ijh}}x_{ijh}\right] \qquad .(25)$$

The sample parameters estimate the following aggregates of population parameters:

$$\underset{123}{E}\ s_1^2\left[\frac{p_i}{\pi_i l_i}\underset{2}{S}\frac{p_{ij}}{\pi_{ij}n_{ij}}\underset{3}{S}\frac{p_{ijh}x_{ijh}}{\pi_{ijh}}\right]$$

$$= \frac{\kappa}{\kappa-\epsilon}\sigma_1^2\left[\frac{p_i}{\pi_i}\sum_2 p_{ij}\sum_3 p_{ijh}\xi_{ijh}\right]$$

$$+\sum_1 \pi_i\frac{\lambda_i-\epsilon_i l_i}{\lambda_i-\epsilon_i}\cdot\frac{1}{l_i}\sigma_2^2\left[\frac{p_i}{\pi_i}\cdot\frac{p_{ij}}{\pi_{ij}}\sum_3 p_{ijh}\xi_{ijh}\right]$$

$$+\sum_1 \pi_i\cdot\frac{1}{l_i}\sum_2 \pi_{ij}\frac{\nu_{ij}-\epsilon_{ij}n_{ij}}{\nu_{ij}-\epsilon_{ij}}\frac{1}{n_{ij}}\sigma_3^2$$

$$\left.\times\left[\frac{p_i}{\pi_i}\cdot\frac{p_{ij}}{\pi_{ij}}\frac{p_{ijh}}{\pi_{ijh}}\xi_{ijh}\right]\right\}$$

$$\underset{23}{E}\, s_2{}^2 \left[\frac{p_i}{\pi_i} \cdot \frac{p_{ij}}{\pi_{ij} n_{ij}} \underset{3}{S} \frac{p_{ijh} x_{ijh}}{\pi_{ijh}}\right]$$

$$= \frac{\lambda_i}{\lambda_i - \epsilon_i} \sigma_2{}^2 \left[\frac{p_i}{\pi_i} \cdot \frac{p_{ij}}{\pi_{ij}} \sum_3 p_{ijh} \xi_{ijh}\right]$$

$$+ \sum_2 \pi_{ij} \frac{\nu_{ij} - \epsilon_{ij} n_{ij}}{\nu_{ij} - \epsilon_{ij}} \cdot \frac{1}{n_{ij}} \sigma_3{}^2 \left[\frac{p_i}{\pi_i} \cdot \frac{p_{ij}}{\pi_{ij}} \cdot \frac{p_{ijh}}{\pi_{ijh}} \xi_{ijh}\right]$$

$$\underset{3}{E}\, s_3{}^2 \left[\frac{p_i}{\pi_i} \cdot \frac{p_{ij}}{\pi_{ij}} \cdot \frac{p_{ijh}}{\pi_{ijh}} x_{ijh}\right]$$

$$= \frac{\nu_{ij}}{\nu_{ij} - \epsilon_{ij}} \sigma_3{}^2 \left[\frac{p_i}{\pi_i} \cdot \frac{p_{ij}}{\pi_{ij}} \cdot \frac{p_{ijh}}{\pi_{ijh}} \xi_{ijh}\right] \tag{26}$$

Finally for a sample of another size:

$$\mathcal{E}\,\mathrm{Var}'(X)$$

$$= \mathcal{E}\,\mathrm{Var}(X) + N^2 \left\{\left(\frac{1}{k'} - \frac{1}{k}\right) s_1{}^2 \left[\frac{p_i}{\pi_i l_i} \underset{2}{S} \frac{p_{ij}}{\pi_{ij} n_{ij}} \underset{3}{S} \frac{p_{ijh} x_{ijh}}{\pi_{ijh}}\right]\right.$$

$$+ \frac{1}{k'k} \underset{1}{S} \left(\frac{1}{l_i'} - \frac{1}{l_i}\right) s_2{}^2 \left[\frac{p_i}{\pi_i} \cdot \frac{p_{ij}}{\pi_{ij} n_{ij}} \underset{3}{S} \frac{p_{ijh} x_{ijh}}{\pi_{ijh}}\right]$$

$$\left. + \frac{1}{k'k} \underset{1}{S} \frac{1}{l_i' l_i} \underset{2}{S} \left(\frac{1}{n'_{ij}} - \frac{1}{n_{ij}}\right) s_3{}^2 \left[\frac{p_i}{\pi_i} \cdot \frac{p_{ij}}{\pi_{ij}} \cdot \frac{p_{ijh}}{\pi_{ijh}} x_{ijh}\right]\right\} \tag{27}$$

This formula is only true if in the new sample the system of constants ϵ, ϵ_i, ϵ_{ij} stays the same.

It can easily be seen, if the selection with varying probabilities is made proportionate to the size of every sampling unit (considering as size the number of tertiary units in every sampling unit), then

$$\pi_i = p_i,\ \pi_{ij} = p_{ij},\ \pi_{ijh} = p_{ijh} \tag{28}$$

and many of the factors in formulæ (20) to (27) reduce to unity. For the case with equal probabilities we have to put

$$\epsilon = \epsilon_i = \epsilon_{ij} = 1;\ \ \pi_i = \frac{1}{K},\ \pi_{ij} = \frac{1}{L_i},\ \pi_{ijh} = \frac{1}{N_{ij}},$$

$$\kappa = K,\ \lambda_i = L_i,\ \nu_{ij} = N_{ij} \tag{29}$$

and we fall back to the formulæ in the second chapter.

These results can be generalized as follows:

Under the conditions of selection described before, the estimator defined in every possible sample:

$$x = a \underset{1}{S} b_i \underset{2}{S} c_{ij} \underset{3}{S} d_{ijh} x_{ijh} \tag{30}$$

is an unbiassed estimate of:

$$Ex = ak \sum_1 \pi_i b_i l_i \sum_2 \pi_{ij} c_{ij} n_{ij} \sum_3 \pi_{ijh} d_{ijh} \xi_{ijh} \tag{31}$$

In this way x is an unbiassed estimate of Ξ only if it is identically fulfilled

$$ak \sum_1 \pi_i b_i l_i \sum_2 \pi_{ij} c_{ij} n_{ij} \sum_3 \pi_{ijh} d_{ijh} \xi_{ijh} = \sum_1 \sum_2 \sum_3 \xi_{ijh} \tag{32}$$

and x is an unbiassed estimator of $\bar{\xi}$ only if it is identically fulfilled

$$ak \sum_1 \pi_i b_i l_i \sum_2 \pi_{ij} c_{ij} n_{ij} \sum_3 \pi_{ijh} d_{ijh} \xi_{ijh} = \frac{1}{N} \sum_1 \sum_2 \sum_3 \xi_{ijh} \tag{33}$$

In all other cases x is an unbiassed estimate of some other parameter of the universe, and thus it can be considered as a biassed estimate of the universe total or mean per tertiary unit.

Formulæ (32) and (33) can be used in the following ways:

(i) for a given sample size defined by k, l_i, n_{ij} and for given probabilities π_i, π_{ij}, π_{ijh} an unbiassed estimate of X should be formed. Then we get from (32):

$$a = \frac{1}{k}, \quad b_i = \frac{1}{\pi_i l_i}, \quad c_{ij} = \frac{1}{\pi_{ij} n_{ij}}, \quad d_{ijh} = \frac{1}{\pi_{ijh}} \tag{34}$$

and we come back to (20).

(ii) for given probabilities π_i, π_{ij}, π_{ijh} and arbitrary a, b_i, c_{ij}, d_{ijh} the sample size k, l_i, n_{ij} should be determined for which (30) gives an unbiassed estimate of X. As (32) imposes only one restriction on k, l_i, n_{ij} many solutions are possible.

(iii) for given values a, b_i, c_{ij}, d_{ijh}, and a given sample size, k, l_i, n_{ij}, the probabilities π_i, π_{ij}, π_{ijh} can be sought which will render (30) an unbiassed estimate of X.

In this paper we will work with the assumption that the sample size k, l_i, n_{ij} and probabilities π_i, π_{ij}, π_{ijh} are fixed in advance. Then (30) with arbitrary a, b_i, c_{ij}, d_{ijh} is generally a biassed estimate of (1). The bias amounts to

$$\Delta = ak \sum_1 \pi_i b_i l_i \sum_2 \pi_{ij} c_{ij} n_{ij} \sum_3 \pi_{ijh} d_{ijh} \xi_{ijh} - \sum_1 \sum_2 \sum_3 \xi_{ijh} \tag{35}$$

The variance error of the estimator x in (30) is given by:

$$\begin{aligned}
\text{Var}(x) &= E(x - Ex)^2 \\
&= \frac{1}{k} \frac{\kappa - \epsilon k}{\kappa - \epsilon} \sigma_1^2 \left[ak \cdot b_i l_i \sum_2 \pi_{ij} c_{ij} n_{ij} \sum_3 \pi_{ijh} d_{ijh} \xi_{ijh} \right] \\
&\quad + \frac{1}{k} \sum_1 \pi_i \frac{\lambda_i - \epsilon_i l_i}{\lambda_i - \epsilon_i} \cdot \frac{1}{l_i} \sigma_2^2 \left[ak \cdot b_i l_i \cdot c_{ij} n_{ij} \sum_3 \pi_{ijh} d_{ijh} \xi_{ijh} \right] \\
&\quad + \frac{1}{k} \sum_1 \pi_i \cdot \frac{1}{l_i} \sum_2 \pi_{ij} \frac{\nu_{ij} - \epsilon_{ij} n_{ij}}{\nu_{ij} - \epsilon_{ij}} \cdot \frac{1}{n_{ij}} \\
&\quad \times \sigma_3^2 \left[ak \cdot b_i l_i \cdot c_{ij} n_{ij} d_{ijh} \xi_{ijh} \right]
\end{aligned} \tag{36}$$

where the population parameters σ_r^2 are defined as in (23′).

We are mostly interested in the mean square, being,

$$M.S.(x) = E(x - \Xi)^2 = \text{Var}(x) + \Delta^2 \tag{37}$$

If a system of constants a, b_i, c_{ij}, d_{ijh}, gives a bias Δ which is negligible compared with the variance error, and if this variance error is sensibly smaller than with other constants which are determined from (32), then such a biassed estimate is to be preferred to the unbiassed estimate (20) or (22).

The estimated bias is

$$\delta = \mathcal{E}\Delta = a \underset{1}{S} b_i \underset{2}{S} c_{ij} \underset{3}{S} d_{ijh} x_{ijh} - \frac{1}{k} \underset{1}{S} \frac{1}{\pi_i l_i} \underset{2}{S} \frac{1}{\pi_{ij} n_{ij}} \underset{3}{S} \frac{x_{ijh}}{\pi_{ijh}} \tag{38}$$

and the estimated variance error is:

$$\begin{aligned}
\mathcal{E}\,\text{Var}(x) &= \frac{1}{k} \left(1 - \frac{\epsilon k}{x}\right) s_1^2 \left[ak \cdot b_i \underset{2}{S} c_{ij} \underset{3}{S} d_{ijh} x_{ijh} \right] \\
&\quad + \frac{\epsilon}{k\kappa} \underset{1}{S} \frac{1}{l_i} \left(1 - \frac{\epsilon_i l_i}{\lambda_i}\right) s_2^2 \left[ak \cdot b_i l_i \cdot c_{ij} \underset{3}{S} d_{ijh} x_{ijh} \right] \\
&\quad + \frac{\epsilon}{k\kappa} \underset{1}{S} \frac{\epsilon_i}{l_i \lambda_i} \underset{2}{S} \frac{1}{n_{ij}} \left(1 - \frac{\epsilon_{ij} n_{ij}}{\nu_{ij}}\right) \\
&\quad \times s_3^2 \left[ak \cdot b_i l_i \cdot c_{ij} n_{ij} \cdot d_{ijh} x_{ijh} \right]
\end{aligned} \tag{39}$$

where the sample parameters s_r^2 are defined as in (24′).

The universe parameters σ_r^2 in (36), can be estimated by estimators

$$\underset{123}{\mathcal{E}}\ \sigma_1^2 \left[ak\cdot b_i l_i \sum_2 \pi_{ij} c_{ij} n_{ij} \sum_3 \pi_{ijh} d_{ijh} \xi_{ijh} \right]$$

$$= \frac{\kappa - \epsilon}{\kappa} \Bigg\{ s_1^2 \left[ak \cdot b_i \underset{2}{S}\, c_{ij} \underset{3}{S}\, d_{ijh} x_{ijh} \right]$$

$$- \frac{1}{k} \underset{1}{S} \frac{1}{l_i} \left(1 - \frac{\epsilon_i l_i}{\lambda_i} \right) s_2^2 \left[ak\cdot b_i l_i \cdot c_{ij} \underset{3}{S}\, d_{ijh} x_{ijh} \right]$$

$$- \frac{1}{k} \underset{1}{S} \frac{\epsilon_i}{l_i \lambda_i} \underset{2}{S} \frac{1}{n_{ij}} \left(1 - \frac{\epsilon_{ij} n_{ij}}{\nu_{ij}} \right)$$

$$\times\ s_3^2 \left[ak \cdot b_i l_i \cdot c_{ij} n_{ij} d_{ijh} x_{ijh} \right] \Bigg\}$$

$$\underset{23}{\mathcal{E}}\ \sigma_2^2 \left[ak \cdot b_i l_i \cdot c_{ij} n_{ij} \sum_3 \pi_{ijh} d_{ijh} \xi_{ijh} \right]$$

$$= \frac{\lambda_i - \epsilon_i}{\lambda_i} \Bigg\{ s_2^2 \left[ak \cdot b_i l_i \cdot c_{ij} \underset{3}{S}\, d_{ijh} x_{ijh} \right]$$

$$- \frac{1}{l_i} \underset{2}{S} \frac{1}{n_{ij}} \left(1 - \frac{\epsilon_{ij} n_{ij}}{\nu_{ij}} \right) s_3^2 \left[ak\cdot b_i l_i \cdot c_{ij} n_{ij} \cdot d_{ijh} x_{ijh} \right] \Bigg\}$$

$$\underset{3}{\mathcal{E}}\ \sigma_3^2 \left[ak\cdot b_i l_i \cdot c_{ij} n_{ij} \cdot d_{ijh} \xi_{ijh} \right]$$

$$= \frac{\nu_{ij} - \epsilon_{ij}}{\nu_{ij}} s_3^2 \left[ak\cdot b_i l_i \cdot c_{ij} n_{ij} \cdot d_{ijh} x_{ijh} \right] \tag{40}$$

The meaning of the symbols σ_r^2 and s_r^2 is the same as (23′) and (24′). The sample parameters, s_r^2, have these expectations:

$$\underset{123}{E}\ s_1^2 \left[ak\cdot b_i \underset{2}{S}\, c_{ij} \underset{3}{S}\, d_{ijh} x_{ijh} \right]$$

$$= \frac{\kappa}{\kappa - \epsilon} \sigma_1^2 \left[ak\cdot b_i l_i \sum_2 \pi_{ij} c_{ij} n_{ij} \sum_3 \pi_{ijh} d_{ijh} \xi_{ijh} \right]$$

$$+ \sum_1 \pi_i \frac{\lambda_i - \epsilon_i l_i}{\lambda_i - \epsilon_i} \cdot \frac{1}{l_i} \sigma_2^2 \left[ak\cdot b_i l_i \cdot c_{ij} n_{ij} \sum_3 \pi_{ijh} d_{ijh} \xi_{ijh} \right]$$

$$+ \sum_1 \pi_i \cdot \frac{1}{l_i} \sum_2 \pi_{ij} \frac{\nu_{ij} - \epsilon_{ij} n_{ij}}{\nu_{ij} - \epsilon_{ij}} \cdot \frac{1}{n_{ij}}$$

$$\times\ \sigma_3^2 \left[ak\cdot b_i l_i,\ c_{ij} n_{ij} \cdot d_{ijh} \xi_{ijh} \right]$$

$$\underset{23}{E}\, s_2^2 \left[ak \cdot b_i l_i,\ c_{ij} \underset{3}{S}\, d_{ijh} x_{ijh} \right]$$

$$= \frac{\lambda_i}{\lambda_i - \epsilon_i} \sigma_2^2 \left[ak \cdot b_i l_i \cdot c_{ij} n_{ij} \sum_3 \pi_{ijh} d_{ijh} \xi_{ijh} \right]$$

$$+ \sum_2 \pi_{ij} \frac{\nu_{ij} - \epsilon_{ij} n_{ij}}{\nu_{ij} - \epsilon_{ij}} \cdot \frac{1}{n_{ij}} \sigma_3^2 \left[ak \cdot b_i l_i \cdot c_{ij} n_{ij} \cdot d_{ijh} \xi_{ijh} \right]$$

$$\underset{3}{E}\, s_3^2 \left[ak \cdot b_i l_i \cdot c_{ij} n_{ij} \cdot d_{ijh} x_{ijh} \right]$$

$$= \frac{\nu_{ij}}{\nu_{ij} - \epsilon_{ij}} \sigma_3^2 \left[ak \cdot b_i l_i \cdot c_{ij} n_{ij} \cdot d_{ijh} \xi_{ijh} \right] \tag{41}$$

For a different sample size we have:

$$\mathcal{E}\, \mathrm{Var}'(x) = \mathcal{E}\, \mathrm{Var}(x) + \left(\frac{1}{k'} - \frac{1}{k} \right) s_1^2 \left[ak \cdot b_i \underset{2}{S}\, c_{ij} \underset{3}{S}\, d_{ijh} x_{ijh} \right]$$

$$+ \frac{1}{k'k} \underset{1}{S} \left(\frac{1}{l_i'} - \frac{1}{l_i} \right) s_2^2 \left[ak \cdot b_i l_i \cdot c_{ij} \underset{3}{S}\, d_{ijh} x_{ijh} \right]$$

$$+ \frac{1}{k'k} \underset{1}{S} \frac{1}{l_i' l_i} \underset{2}{S} \left(\frac{1}{n_{ij}'} - \frac{1}{n_{ij}} \right)$$

$$\times\, s_3^2 \left[ak \cdot b_i l_i \cdot c_{ij} n_{ij} \cdot d_{ijh} x_{ijh} \right] \tag{42}$$

4. Mathematical Derivations

The case of selection with varying probabilities includes the case of selection with equal probabilities as a special case. However the reverse is also true; the formulæ (20) to (27) can easily be developed from the formulæ (3) to (19) in the following way:—

Let us suppose that we have the universe described in Chapter 2, where in the case of selection with equal probabilities formulæ (3) to (19) apply. To simplify the derivation let us first suppose that the selection in each stage is the common selection without replacement, *i.e.*, $\epsilon = \epsilon_i = \epsilon_{ij} = 1$. Instead of changing the probabilities, we will change the universe in the following way: Formerly in the i-th primary unit, j-th secondary unit, there were N_{ij} tertiary units with the value of the observed character being ξ_{ijh}; now let us expand the universe so that every former tertiary unit is repeated ν_{ijh} times, making a total of $\sum_{h=1}^{N_{ij}} \nu_{ijh} = \nu_{ij}$ tertiary units in the j-th secondary unit of the i-th primary unit. After this every newly formed secondary unit in the i-th primary unit (earlier there were in it L_i secondary units) should be repeated

λ_{ij} times, making a total of $\sum_{j=1}^{L_i} \lambda_{ij} = \lambda_i$ secondary units in the i-th primary unit. Finally every newly formed i-th primary unit should be repeated κ_i times, making a total of $\sum_{i=1}^{K} \kappa_i = \kappa$ primary units. If we select from such a universe k, l_i, n_{ij}, primary, secondary and tertiary sampling units respectively *with equal probabilities*, this selection is equivalent to a selection of the same number of units from the initial universe with the varying probabilities being $\pi_i = \kappa_i/\kappa$ in the first stage, $\pi_{ij} = \lambda_{ij}/\lambda_i$ in the second stage and $\pi_{ijh} = \nu_{ijh}/\nu_{ij}$ in the third stage. This means that for this expanded universe the formulæ (3) to (19) apply, yielding after some transformations formulæ for a selection with varying probabilities from the initial universe.

In expanding the universe we have made a change in its total and average per tertiary unit. To counterbalance this in the case of the universe total, we must modify simultaneously the values of the observed character, ξ_{ijh}'. It turns out that the convenient modifications of these values is obtained by applying the formula:

$$\zeta_{ijh} = \frac{\xi_{ijh}}{\kappa_i \lambda_{ij} \nu_{ijh}} \tag{43}$$

In the same way the values in the sample should be transformed by

$$z_{ijh} = \frac{x_{ijh}}{\kappa_i \lambda_{ij} \nu_{ijh}} \tag{43'}$$

The transformation for the case of the universe mean will not be studied separately, because when formulæ for the universe total are developed it is easy to modify them for the estimates of the universe mean per tertiary unit.

Thus to arrive at the definitive formulæ we will follow these steps:

(i) Rewrite formulæ (3) to (19) making the following substitutions:

Formerly ..	ξ_{ijh}	x_{ijh}	N_{ij}	L_i	K	$\sum_1$	$\sum_2$	$\sum_3$
Now.. ..	ζ_{ijh}	z_{ijh}	ν_{ij}	λ_i	κ	$\sum_{i=1}^{\kappa}$	$\sum_{j=1}^{\lambda_i}$	$\sum_{h=1}^{\nu_{ij}}$

(44)

(ii) Substitute new values for the observed character mentioned in (43) and (43'). This is to achieve the result that the universe total for the new universe turns out to be the same as for the initial universe. Indeed:

$$\sum_{i=1}^{\kappa} \sum_{j=1}^{\lambda_i} \sum_{h=1}^{\nu_{ij}} \zeta_{ijh} = \sum_{i=1}^{K} \kappa_i \sum_{j=1}^{L_i} \lambda_{ij} \sum_{h=1}^{N_{ij}} \nu_{ijh} \zeta_{ijh} = \sum_1 \sum_2 \sum_3 \xi_{ijh} \tag{45}$$

So both universes, the enlarged one and the initial one, are made equivalent, but what for the first one is selection with equal probabilities, for the second one is selection with probabilities π_i, π_{ij}, π_{ijh} in the respective stages.

(iii) We arrange to come back to formulæ containing only summations $\sum_{i=1}^{K}$, $\sum_{j=1}^{L_i}$, $\sum_{h=1}^{N_{ij}}$ by using the following obvious conversions of summations:

$$\sum_{i=1}^{\kappa} t = \sum_{i=1}^{K} \chi_i t; \quad \sum_{j=1}^{\lambda_i} t = \sum_{j=1}^{L_i} \lambda_{ij} t; \quad \sum_{h=1}^{\nu_{ij}} t = \sum_{h=1}^{N_{ij}} \nu_{ijh} t \tag{46}$$

Let us demonstrate these steps on formulæ (3) yielding a unbiassed estimate of X. Using (44) [step (i)] we rewrite formula (3) as:

$$\chi = \frac{\kappa}{k} \underset{1}{S} \frac{\lambda_i}{l_i} \underset{2}{S} \frac{\nu_{ij}}{n_{ij}} \underset{3}{S} z_{ijh}$$

The use of (43) [step (ii)] brings us to:

$$\chi = \frac{\kappa}{k} \underset{1}{S} \frac{\lambda_i}{l_i} \underset{2}{S} \frac{\nu_{ij}}{n_{ij}} \underset{3}{S} \frac{x_{ijh}}{\kappa_i \lambda_{ij} \nu_{ijh}} = \frac{1}{k} \underset{1}{S} \frac{1}{\pi_i l_i} \underset{2}{S} \frac{1}{\pi_{ij} n_{ij}} \underset{3}{S} \frac{x_{ijh}}{\pi_{ijh}}$$

where we have inserted:

$$\pi_i = \frac{\kappa_i}{\kappa}, \quad \pi_{ij} = \frac{\lambda_{ij}}{\lambda_i}, \quad \pi_{ijh} = \frac{\nu_{ijh}}{\nu_{ij}}.$$

Indeed this brings us exactly to formula (20).

For another demonstration we write first, in a similar way using (44), formula (7) in the form:

$$\begin{aligned} \mathrm{Var}\,(\chi) = \left(\frac{\kappa}{k}\right)^2 \Bigg\{ & k \cdot \frac{\kappa - k}{\kappa - 1} \sigma_1^2 \left[\sum_{j=1}^{\lambda_i} \sum_{h=1}^{\nu_{ij}} \zeta_{ijh} \right] \\ & + \frac{k}{\kappa} \sum_{i=1}^{\kappa} \frac{\lambda_i - l_i}{\lambda_i - 1} \cdot \left(\frac{\lambda_i}{l_i}\right)^2 \cdot l_i \sigma_2^2 \left[\sum_{h=1}^{\nu_{ij}} \zeta_{ijh} \right] \\ & + \frac{k}{\kappa} \sum_{i=1}^{\kappa} \left(\frac{\lambda_i}{l_i}\right)^2 \cdot l_i \cdot \frac{1}{\lambda_i} \sum_{j=1}^{\lambda_i} \frac{\nu_{ij} - n_{ij}}{\nu_{ij} - 1} \\ & \times \left(\frac{\nu_{ij}}{n_{ij}}\right)^2 \cdot n_{ij} \sigma_3^2 \left[\zeta_{ijh}\right] \Bigg\} \end{aligned} \tag{47}$$

and after substituting (43) we have

$$\sum_{j=1}^{\lambda_i} \sum_{h=1}^{\nu_{ij}} \zeta_{ijh} = \sum_{j=1}^{L_i} \lambda_{ij} \sum_{h=1}^{N_{ij}} \nu_{ijh} \cdot \frac{\xi_{ijh}}{\kappa_i \lambda_{ij} \nu_{ijh}} = \frac{1}{\kappa_i} \sum_2 \sum_3 \xi_{ijh}$$

$$\sum_{h=1}^{\nu_{ij}} \zeta_{ijh} = \qquad = \frac{1}{\kappa_i \lambda_{ij}} \sum_3 \xi_{ijh}$$

$$\zeta_{ijh} = \qquad = \frac{1}{\kappa_i \lambda_{ij} \nu_{ijh}} \xi_{ijh}$$

and having in mind that because of (1): $N p_i p_{ij} p_{ijh} = 1$, further using also (46) and putting in the first term of (47) κ^2, in the second term $\kappa^2 \lambda_i^2$, in the third $\kappa^2 \lambda_i^2 \nu_{ij}^2$ under the σ^2 sign, we obtain finally

$$\begin{aligned} \operatorname{Var}(X) = N^2 \Bigg\{ & \frac{1}{k} \frac{\kappa - k}{\kappa - 1} \sigma_1^2 \left[\frac{p_i}{\pi_i} \sum_2 p_{ij} \sum_3 p_{ijh} \xi_{ijh} \right] \\ & + \frac{1}{k} \sum_{i=1}^{K} \pi_i \frac{\lambda_i - l_i}{\lambda_i - 1} \cdot \frac{1}{l_i} \sigma_2^2 \left[\frac{p_i}{\pi_i} \cdot \frac{p_{ij}}{\pi_{ij}} \sum_3 p_{ijh} \xi_{ijh} \right] \\ & + \frac{1}{k} \sum_{i=1}^{K} \pi_i \cdot \frac{1}{l_i} \sum_{j=1}^{L_i} \pi_{ij} \frac{\nu_{ij} - n_{ij}}{\nu_{ij} - 1} \cdot \frac{1}{n_{ij}} \\ & \times \sigma_3^2 \left[\frac{p_i}{\pi_i} \cdot \frac{p_{ij}}{\pi_{ij}} \cdot \frac{p_{ijh}}{\pi_{ijh}} \xi_{ijh} \right] \Bigg\} \end{aligned} \tag{48}$$

The meaning of the universe parameters σ^2 in (48) has also changed compared with (10). Starting from (10) we have now from (47)

$$\sigma_1^2 [t] = \frac{1}{\kappa} \sum_{i=1}^{\kappa} \left(t - \frac{1}{\kappa} \sum_{i=1}^{\kappa} t \right)^2 = \frac{1}{\kappa} \sum_{i=1}^{K} \kappa_i \left(t - \frac{1}{\kappa} \sum_{i=1}^{K} \kappa_i t \right)^2$$

$$= \sum_{i=1}^{K} \pi_i \left(t - \sum_{i=1}^{K} \pi_i t \right)^2 \tag{48'}$$

and similarly:

$$\sigma_2^2 [t] = \sum_{j=1}^{L_i} \pi_{ij} \left(t - \sum_{j=1}^{L_i} \pi_{ij} t \right)^2; \quad \sigma_3^2 [t] = \sum_{h=1}^{N_{ij}} \pi_{ijh} \left(t - \sum_{h=1}^{N_{ij}} \pi_{ijh} t \right)^2$$

Formula (48) is almost the same as (23) except for the missing factors ϵ, ϵ_i, ϵ_{ij}. However this case is now easy to master too. We have only to make a contraction of the earlier enlarged universe.[5] Instead of

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[5] The reader is warned not to do this contraction after the formula (48) is developed, because this creates certain difficulties.

introducing ν_{ijh} tertiary units where there was earlier one, we introduce ν_{ijh}/ϵ_{ijh}. In the same way, the secondary units are repeated λ_{ij}/ϵ_i times, the primary units κ_i/ϵ times. Then a selection with equal probabilities from such a universe is equivalent to a selection from the original universe with varying probabilities π_i, π_{ij}, π_{ijh} using a system of non-replacement with constants ϵ, ϵ_i, ϵ_{ij}. This could be achieved if we had immediately made substitutions:

Formerly..	ξ_{ijh}	x_{ijh}	N_{ij}	L_i	K	$\sum\limits_{1}$	$\sum\limits_{2}$	$\sum\limits_{3}$
Now ..	ζ_{ijh}	z_{ijh}	ν_{ij}/ϵ_{ij}	λ_i/ϵ_i	κ/ϵ	$\sum\limits_{i=1}^{\chi/\epsilon}$	$\sum\limits_{j=1}^{\lambda_i/\epsilon_i}$	$\sum\limits_{h=1}^{\nu_{ij}/\epsilon_{ij}}$

(49)

putting afterwards

$$\zeta_{ijh} = \frac{\epsilon\epsilon_i\epsilon_{ij}\xi_{ijh}}{\kappa_i\lambda_{ij}\nu_{ijh}} \quad \text{and} \quad z_{ijh} = \frac{\epsilon\epsilon_i\epsilon_{ij}x_{ijh}}{\kappa_i\lambda_{ij}\nu_{ijh}} \tag{50}$$

and reducing the summations in this way:

$$\sum_{i=1}^{\kappa/\epsilon} t = \frac{1}{\epsilon}\sum_{i=1}^{K}\kappa_i t;\quad \sum_{j=1}^{\lambda_i/\epsilon_i} t = \frac{1}{\epsilon_i}\sum_{j=1}^{L_i}\lambda_{ij}t_j \quad \sum_{h=1}^{\nu_{ij}/\epsilon_{ij}} t = \frac{1}{\epsilon_{ij}}\sum_{h=1}^{N_{ij}}\nu_{ijh}t \tag{51}$$

This gives immediately from (3)

$$\begin{aligned}\chi &= \frac{x}{\epsilon k}\underset{1}{S}\frac{\lambda_i}{\epsilon_i l_i}\underset{2}{S}\frac{\nu_{ij}}{\epsilon_{ij}n_{ij}}\underset{3}{S}\frac{\epsilon\epsilon_i\epsilon_{ij}x_{ijh}}{x_i\lambda_{ij}\nu_{ijh}}\\ &= \frac{1}{k}\underset{1}{S}\frac{1}{\pi_i l_i}\underset{2}{S}\frac{1}{\pi_{ij}n_{ij}}\underset{3}{S}\frac{x_{ijh}}{\pi_{ijh}}\end{aligned}$$

and so formula (20) did not change. But from (7) we obtain

$$\begin{aligned}\operatorname{Var}(X) = \left(\frac{\kappa}{\epsilon k}\right)^2\cdot\Bigg\{&k\,\frac{\kappa-\epsilon k}{\kappa-\epsilon}\,\sigma_1{}^2\left[\sum_{j=1}^{\lambda_i/\epsilon_i}\sum_{h=1}^{\nu_{ij}/\epsilon_{ij}}\zeta_{ijh}\right]\\ &+\frac{k\epsilon}{\kappa}\sum_{i=1}^{\kappa/\epsilon}\frac{\lambda_i-\epsilon_i l_i}{\lambda_i-\epsilon_i}\cdot\left(\frac{\lambda_i}{\epsilon_i l_i}\right)^2 l_i\sigma_2{}^2\left[\sum_{h=1}^{\nu_{ij}/\epsilon_{ij}}\zeta_{ijh}\right]\\ &+\frac{k\epsilon}{\kappa}\sum_{i=1}^{\kappa/\epsilon}\left(\frac{\lambda_i}{\epsilon_i l_i}\right)^2\cdot l_i\cdot\frac{\epsilon_i}{\lambda_i}\sum_{j=1}^{\lambda_i/\epsilon_i}\frac{\nu_{ij}-\epsilon_{ij}n_{ij}}{\nu_{ij}-\epsilon_{ij}}\\ &\times\left(\frac{\nu_{ij}}{\epsilon_{ij}n_{ij}}\right)^2\cdot n_{ij}\sigma_3{}^2\left[\zeta_{ijh}\right]\Bigg\}\end{aligned}$$

Further we take into account that:

$$\sum_{j=1}^{\lambda_i/\epsilon} \sum_{h=1}^{\nu_{ij}/\epsilon_i} \zeta_{ijh} = \frac{1}{\epsilon_i} \sum_{j=1}^{L_i} \lambda_{ij} \frac{1}{\epsilon_{ij}} \sum_{h=1}^{N_{ij}} \nu_{ijh} \cdot \frac{\epsilon\,\epsilon_i\,\epsilon_{ij}\,\xi_{ijh}}{\kappa_i \lambda_{ij} \nu_{ijh}}$$

$$= \frac{\epsilon}{\kappa_i} \sum_{j=1}^{L_i} \sum_{h=1}^{N_{ij}} \xi_{ijh}$$

$$\sum_{h=1}^{\nu_{ij}/\epsilon_{ij}} \zeta_{ijh} = \frac{\epsilon}{\kappa_i} \cdot \frac{\epsilon_i}{\lambda_{ij}} \sum_{h=1}^{N_{ij}} \xi_{ijh} \qquad (52)$$

and that in the formula above for Var (X)

$$\sigma_1^2\,[t] = \frac{\epsilon}{\kappa} \sum_{i=1}^{\kappa/\epsilon} \left(t - \frac{\epsilon}{\kappa} \sum_{i=1}^{\kappa/\epsilon} t \right)^2 = \sum_{i=1}^{K} \frac{\kappa_i}{\kappa} \left(t - \sum_{i=1}^{K} \frac{\kappa_i}{\kappa}\, t \right)^2$$

$$= \sum_{i=1}^{K} \pi_i \left(t - \sum_{i=1}^{K} \pi_i t \right)^2 \qquad (53)$$

and similarly for σ_2^2 and σ_3^2.

After this we multiply the right side of Var (X) with $Np_i p_{ij} p_{ijh} = 1$ which enters into the brackets of σ_1^2, σ_2^2 and σ_3^2 and so we obtain finally (23).

Another demonstration.—Let us apply now (49), (50) and (51) to formula (12). We get first:

$$\mathcal{E}\ \text{Var}\,(\chi) = \left(\frac{\kappa}{\epsilon k}\right)^2 \cdot \left\{ k\left(1 - \frac{\epsilon k}{\kappa}\right) s_1^2 \left[\frac{\lambda_i}{\epsilon_i l_i} \mathop{S}_{2} \frac{\nu_{ij}}{\epsilon_{ij} n_{ij}} \mathop{S}_{3} z_{ijh} \right] \right.$$

$$+ \frac{k\epsilon}{\kappa} \mathop{S}_{1} \left(1 - \frac{\epsilon_i l_i}{\lambda_i}\right) \cdot \left(\frac{\lambda_i}{\epsilon_i l_i}\right)^2 \cdot l_i s_2^2 \left[\frac{\nu_{ij}}{\epsilon_{ij} n_{ij}} \mathop{S}_{3} z_{ijh} \right]$$

$$+ \frac{k\epsilon}{\kappa} \mathop{S}_{1} \left(\frac{\lambda_i}{\epsilon_i l_i}\right)^2 \cdot l_i \cdot \frac{\epsilon_i}{\lambda_i} \mathop{S}_{2} \left(1 - \frac{\epsilon_{ij} n_{ij}}{\nu_{ij}}\right)$$

$$\left. \times \left(\frac{\nu_{ij}}{\epsilon_{ij} n_{ij}}\right)^2 \cdot n_{ij} s_3^2 \ [z_{ijh}] \right\}$$

and when we apply (50):

$$\frac{\lambda_i}{\epsilon_i l_i} \mathop{S}_{2} \frac{\nu_{ij}}{\epsilon_{ij} n_{ij}} \mathop{S}_{3} z_{ijh} = \frac{\lambda_i}{\epsilon_i l_i} \mathop{S}_{2} \frac{\nu_{ij}}{\epsilon_{ij} n_{ij}} \mathop{S}_{3} \frac{\epsilon\,\epsilon_i\,\epsilon_{ij}\,x_{ijh}}{\kappa_i \lambda_{ij} \nu_{ijh}}$$

$$= \frac{\epsilon}{\kappa_i} \cdot \frac{1}{l_i} \mathop{S}_{2} \frac{1}{\pi_{ij} n_{ij}} \mathop{S}_{3} \frac{x_{ijh}}{\pi_{ijh}}$$

$$\frac{\nu_{ij}}{\epsilon_{ij} n_{ij}} \mathop{S}_{3} z_{ijh} = \frac{\epsilon}{\kappa_i} \cdot \frac{\epsilon_i}{\lambda_{ij}} \cdot \frac{1}{n_{ij}} \mathop{S}_{3} \frac{x_{ijh}}{\pi_{ijh}} \qquad (54)$$

The meaning of s^2 according to (24′) does not change; so we multiply the right side with $Np_i p_{ij} p_{ijh} = 1$, and applying (51) we get finally (24).

For a final demonstration, let us consider the first formula in (16). By applying (49) we get

$$\underset{123}{\mathcal{E}}\ \sigma_1^2 \left[\sum_{j=1}^{\lambda_i/\epsilon_i} \sum_{h=1}^{\nu_{ij}/\epsilon_{ij}} \zeta_{ijh} \right]$$

$$= \frac{\kappa - \epsilon}{\kappa} \left\{ s_1^2 \left[\frac{\lambda_i}{\epsilon_i l_i} \underset{2}{S} \frac{\nu_{ij}}{\epsilon_{ij} n_{ij}} \underset{3}{S}\ z_{ijh} \right] \right.$$

$$- \frac{1}{k} \underset{1}{S} \left(1 - \frac{\epsilon_i l_i}{\lambda_i}\right) \cdot \left(\frac{\lambda_i}{\epsilon_i l_i}\right)^2 \cdot l_i s_2^2 \left[\frac{\nu_{ij}}{\epsilon_{ij} n_{ij}} \underset{3}{S}\ z_{ijh} \right]$$

$$- \frac{1}{k} \underset{1}{S} \left(\frac{\lambda_i}{\epsilon_i l_i}\right)^2 \cdot l_i \cdot \frac{\epsilon_i}{\lambda_i} \underset{2}{S} \left(1 - \frac{\epsilon_{ij} n_{ij}}{\nu_{ij}}\right) \cdot \left(\frac{\nu_{ij}}{\epsilon_{ij} n_{ij}}\right)^2$$

$$\left. \times\ n_{ij} s_3^2\ [z_{ijh}] \right\}$$

When we apply (50) and the relations (52), (53) and (54) we get first

$$\underset{123}{\mathcal{E}}\ \sigma_1^2 \left[\frac{\epsilon}{\kappa_i} \sum_{j=1}^{L_i} \sum_{h=1}^{N_{ij}} \xi_{ijh} \right]$$

$$= \frac{\kappa - \epsilon}{\kappa} \left\{ s_1^2 \left[\frac{\epsilon}{\kappa_i} \cdot \frac{1}{l_i} \underset{2}{S} \frac{1}{\pi_{ij} n_{ij}} \underset{3}{S} \frac{x_{ijh}}{\pi_{ijh}} \right] \right.$$

$$- \frac{1}{k} \underset{1}{S} \left(1 - \frac{\epsilon_i l_i}{\lambda_i}\right) \left(\frac{\lambda_i}{\epsilon_i l_i}\right)^2 \cdot l_i s_2^2 \left[\frac{\epsilon}{\kappa_i} \frac{\epsilon_i}{\lambda_{ij}} \cdot \frac{1}{n_{ij}} \underset{3}{S} \frac{x_{ijh}}{\pi_{ijh}} \right]$$

$$- \frac{1}{k} \underset{1}{S} \left(\frac{\lambda_i}{\epsilon_i l_i}\right)^2 \cdot l_i \cdot \frac{\epsilon_i}{\lambda_i} \underset{2}{S} \left(1 - \frac{\epsilon_{ij} n_{ij}}{\nu_{ij}}\right) \left(\frac{\nu_{ij}}{\epsilon_{ij} n_{ij}}\right)^2$$

$$\left. \times\ n_{ij} s_3^2 \left[\frac{\epsilon}{\kappa_i} \frac{\epsilon_i}{\lambda_{ij}} \frac{\epsilon_{ij}}{\nu_{ijh}}\ x_{ijh} \right] \right\}.$$

If we multiply both sides by $(\kappa/\epsilon)^2$, we get into the brackets of σ_1^2, s_1^2, s_2^2 and s_3^2 a factor $1/\pi_i$. Finally we multiply both sides by $(p_i p_{ij} p_{ijh})^2$ and so we obtain the first result in (25). Others are derived in an analogous way. In this way formulæ (20) to (27) are proved.

To derive formulæ (30) to (42) we start with formula (20), which is an unbiassed estimate of the population total if the selection is made with varying probabilities, and without replacement in which the reduction of chances ϵ, ϵ_i, ϵ_{ij}, in the first, second and third stage is applied.

We substitute instead of x_{ijh} the expression $\alpha\beta_i\gamma_{ij}\delta_{ijh}x_{ijh}$. Then, due to the fact that (20) is an unbiassed estimate of (1), it results that

$$\frac{\alpha}{k} \underset{1}{S} \frac{\beta_i}{\pi_i l_i} \underset{2}{S} \frac{\gamma_{ij}}{\pi_{ij} n_{ij}} \underset{3}{S} \frac{\delta_{ijh} x_{ijh}}{\pi_{ijh}}$$

is an unbiassed estimate of

$$\alpha \sum_1 \beta_i \sum_2 \gamma_{ij} \sum_3 \delta_{ijh} \xi_{ijh}$$

and therefore it is a biassed estimate of either (1) or (2). If we put now

$$\frac{\alpha}{k} = a, \quad \frac{\beta_i}{\pi_i l_i} = b_i, \quad \frac{x_{ij}}{\pi_{ij} n_{ij}} = c_{ij}, \quad \frac{\delta_{ijh}}{\pi_{ijh}} = d_{ijh}$$

the same statement indicates that

$$\chi = a \underset{1}{S} b_i \underset{2}{S} c_{ij} \underset{3}{S} d_{ijh} x_{ijh}$$

where

$$a, \; b_i, \; c_{ij}, \; d_{ijh}$$

are constants in an unbiassed estimate of (31), which was to be proved.

Herefrom the conditions (32) and (33) are obviously imposed if we want to have unbiassed estimates of the universe total or of the mean per tertiary unit.

The most direct proof for further formulæ consists in substituting into the formulæ (23) to (27) which were just proved instead of x_{ijh}, the expression:

$$ak \cdot \pi_i b_i l_i \cdot \pi_{ij} c_{ij} n_{ij} \cdot \pi_{ijh} d_{ijh} x_{ijh}$$

and similarly for ξ_{ijh}. It can be readily seen that this leads to formulæ (36) till (42), which was to be proved.

5. Applications of the Results to Special Cases of Three-Stage Samples

If some of the summations in formula (20) or (30) are complete, then we have special cases of three-stage samples. Let us symbolically represent one of such cases by $S\Sigma S$, which means application of an estimating formula:

$$\chi = a \underset{1}{S} b_i \sum_2 c_{ij} \underset{3}{S} d_{ijh} x_{ijh}.$$

Tnis is a two-stage sample with stratification within primary units (substratification). Then all formulæ in Section 3 have to be changed accordingly. In the following table, special values are given to be inserted in the formulæ.

TABLE I

Special values to be put into formulæ (20) *to* (42) *if the three-stage sample is modified by complete enumerations in some stages*

Symbolical representation of the sample	Name of sample	Special values of constants
SSS	Three-stage sample	..
ΣSS	Stratified two-stage sample	$k=K=\kappa,\ \epsilon=1,\ \pi_i=1/K$
$S\Sigma S$	Substratified two-stage sample	$l_i=L_i=\lambda_i,\ \epsilon_i=1,\ \pi_{ij}=1/L_i$
$SS\Sigma$	Two-stage cluster sampling	$n_{ij}=N_{ij}=\nu_{ij},\ \epsilon_{ij}=1,\ \pi_{ijh}=1/N_{ij}$
$\Sigma\Sigma S$	Two-fold stratified sample	$k=K=\kappa,\ l_i=L_i=\lambda_i,\ \epsilon=\epsilon_i=1,\ \pi_i=1/K,\ \pi_{ij}=1/L_i$
$\Sigma S\Sigma$	Stratified cluster sampling	$k=K=\kappa,\ n_{ij}=N_{ij}=\nu_{ij},\ \epsilon=\epsilon_{ij}=1,\ \pi_i=1/K,\ \pi_{ijh}=1/N_{ij}$
$S\Sigma\Sigma$	Two-fold cluster sampling	$l_i=L_i=\lambda_i,\ n_{ij}=N_{ij}=\nu_{ij},\ \epsilon_i=\epsilon_{ij}=1,\ \pi_{ij}=1/L_i,\ \pi_{ijh}=1/N_{ij}$
$\Sigma\Sigma\Sigma$	Complete enumeration	$k=K=\kappa,\ l_i=L_i=\lambda_i,\ n_{ij}=N_{ij}=\nu_{ij},\ \epsilon=\epsilon_i=\epsilon_{ij}=1,\ \pi_i=1/K,\ \pi_{ij}=1/L_i,\ \pi_{ijh}=1/N_{ij}$

Example.—For a substratified two-stage sample, the unbiassed estimate of the universe total (1) will be (see 20):

$$X=\frac{1}{k}\underset{1}{S}\frac{1}{\pi_i}\sum_{2}\frac{1}{n_{ij}}\underset{3}{S}\frac{x_{ijh}}{\pi_{ijh}}$$

with the variance error (see 23)

$$\text{Var}(X)=N^2\left\{\frac{1}{k}\frac{\kappa-\epsilon k}{\kappa-\epsilon}\sigma_1^2\left[\frac{p_i}{\pi_i}\sum_{2}p_{ij}\sum_{3}p_{ijh}\xi_{ijh}\right]\right.$$

$$\left.+\frac{1}{k}\sum_{1}\pi_i\sum_{2}\frac{\nu_{ij}-\epsilon_{ij}n_{ij}}{\nu_{ij}-\epsilon_{ij}}\cdot\frac{1}{n_{ij}}\sigma_3^2\left[\frac{p_i}{\pi_i}\cdot p_{ij}\cdot\frac{p_{ijh}}{\pi_{ijh}}\xi_{ijh}\right]\right\}$$

and estimated variance error (see 24)

$$\mathcal{E}\,\text{Var}(X) = N^2 \left\{ \frac{1}{k}\left(1 - \frac{\epsilon k}{\kappa}\right) s_1^2 \left[\frac{p_i}{\pi_i} \sum_2 \frac{p_{ij}}{n_{ij}} \underset{3}{S} \frac{p_{ijh} x_{ijh}}{\pi_{ijh}}\right] \right.$$

$$\left. + \frac{\epsilon}{k\kappa} \underset{1}{S} \sum_2 \frac{1}{n_{ij}} \left(1 - \frac{\epsilon_{ij} n_{ij}}{\nu_{ij}}\right) s_3^2 \left[\frac{p_i}{\pi_i} \cdot p_{ij} \cdot \frac{p_{ijh}}{\pi_{ijh}} x_{ijh}\right] \right\}$$

We see that neither the variance error, nor the estimate contain the term concerning the variation of $\xi_{ijh}\,(x_{ijh})$ between secondary units within the primary units.

Further from (25) we see that:

$$\underset{123}{\mathcal{E}}\, \sigma_1^2 \left[\frac{p_i}{\pi_i} \sum_2 p_{ij} \sum_3 p_{ijh} \xi_{ijh}\right]$$

$$= \frac{\kappa - \epsilon}{\kappa} \left\{ s_1^2 \left[\frac{p_i}{\pi_i} \sum_2 \frac{p_{ij}}{n_{ij}} \underset{3}{S} \frac{p_{ijh} x_{ijh}}{\pi_{ijh}}\right] \right.$$

$$\left. - \frac{1}{k} \underset{1}{S} \sum_2 \frac{1}{n_{ij}} \left(1 - \frac{\epsilon_{ij} n_{ij}}{\nu_{ij}}\right) s_3^2 \left[\frac{p_i}{\pi_i} \cdot p_{ij} \cdot \frac{p_{ijh}}{\pi_{ijh}} x_{ijh}\right] \right\}, \text{ etc.}$$

6. Application to Three-Stage Samples Selected with Probability Proportionate to Size

If undersize is considered the number of tertiary units in a primary or secondary unit, then we have to put simply into the formulæ in Chapter 3:

$$\pi_i = \frac{N_i}{N} = p_i, \quad \pi_{ij} = \frac{N_{ij}}{N_i} = p_{ij}, \quad \pi_{ijh} = \frac{1}{N_{ij}} = p_{ijh}.$$

Then, in the general case of such selection,—where the way of selection is characterized by ϵ, ϵ_i, ϵ_{ij}—we obtain from (20) the estimate of the population total:

$$X = \frac{N}{k} \underset{1}{S} \frac{1}{l_i} \underset{2}{S} \frac{1}{n_{ij}} \underset{3}{S}\, x_{ijh}$$

from (23) its variance error:

$$\text{Var}(X) = N^2 \left\{ \frac{1}{k} \frac{\kappa - \epsilon k}{\kappa - \epsilon} \sigma_1^2 \left[\sum_2 p_{ij} \sum_3 p_{ijh} \xi_{ijh}\right] \right.$$

$$+ \frac{1}{k} \sum_1 p_i \frac{\lambda_i - \epsilon_i l_i}{\lambda_i - \epsilon_i} \cdot \frac{1}{l_i} \sigma_2^2 \left[\sum_3 p_{ijh} \xi_{ijh}\right]$$

$$\left. + \frac{1}{k} \sum_1 p_i \cdot \frac{1}{l_i} \sum_2 p_{ij} \frac{\nu_{ij} - \epsilon_{ij} n_{ij}}{\nu_{ij} - \epsilon_{ij}} \cdot \frac{1}{n_{ij}} \sigma_3^2 \left[\xi_{ijh}\right] \right\}$$

$$= N^2 \left\{ \frac{1}{k} \frac{\kappa - \epsilon k}{\kappa - \epsilon} \sigma_1^2 \left[\frac{\sum\limits_2 \sum\limits_3 \xi_{ijh}}{N p_i} \right] \right.$$

$$+ \frac{1}{k} \sum_1 p_i \frac{\lambda_i - \epsilon_i l_i}{\lambda_i - \epsilon_i} \cdot \frac{1}{l_i} \sigma_2^2 \left[\frac{\sum\limits_3 \xi_{ijh}}{N p_i p_{ij}} \right]$$

$$\left. + \frac{1}{k} \sum_1 p_i \cdot \frac{1}{l_i} \sum_2 p_{ij} \frac{\nu_{ij} - \epsilon_{ij} n_{ij}}{\nu_{ij} - \epsilon_{ij}} \cdot \frac{1}{n_{ij}} \sigma_3^2 \left[\frac{\xi_{ijh}}{N p_i p_{ij} p_{ijh}} \right] \right\}$$

from (24) its estimated variance error:

$$\mathcal{E}\,\mathrm{Var}\,(X) = N^2 \left\{ \frac{1}{k} \left(1 - \frac{\epsilon k}{\kappa}\right) s_1^2 \left[\frac{1}{l_i} \underset{2}{S} \frac{1}{n_{ij}} \underset{3}{S}\, x_{ijh} \right] \right.$$

$$+ \frac{\epsilon}{k\kappa} \underset{1}{S} \frac{1}{l_i} \left(1 - \frac{\epsilon_i l_i}{\lambda_i}\right) s_2^2 \left[\frac{1}{n_{ij}} \underset{3}{S}\, x_{ijh} \right]$$

$$\left. + \frac{\epsilon}{k\kappa} \underset{1}{S} \frac{\epsilon_i}{l_i \lambda_i} \underset{2}{S} \frac{1}{n_{ij}} \left(1 - \frac{\epsilon_{ij} n_{ij}}{\nu_{ij}}\right) s_3^2 \; [x_{ijh}] \right\}. \qquad (55)$$

Other formulæ can be obtained in an analogous way from formulæ (25), (26) and (27). If the chances of selection in each stage are made *equal* to the size of the sampling unit, we can insert into these formulæ N, N_i, N_{ij} instead of κ, λ_i, ν_{ij}. The formula for Var (X) was already quoted for a two-stage sample with $\epsilon = 1$, $\epsilon_i = p$, $\kappa = N$, $\lambda_i = N_i$ by Thionet.[6]

If we want to specialize the three-stage sampling in introducing complete summations in some stages, then we must not specialize the formulæ used above, but must first develop general formulæ for the special combination, as in Chapter 5, and after this insert the special values for the probabilities π_i, π_{ij}, π_{ijh}, which are not yet fixed by the sampling design itself.

7. Application to Three-Stage Samples with Varying Probabilities in which in Some Stages Selection with Replacement or with Complete Non-Replacement is Applied

In stages where selection *with replacement* is applied, the corresponding constants ϵ, ϵ_i, ϵ_{ij}, should be put equal to zero. This can already be done with the general formulæ in Chapter 3, but it can also be done with the specialized ones in Chapters 5 and 6.

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[6] See Ref. [6], page 146. This is not in accordance with the results given in a later publication, Ref. [7], page 170.

Example.—The formula for the variance error of the combination $S\Sigma S$ is required where $\epsilon = 0$, $\epsilon_{ij} = 1$ whilst ϵ_i is bound to be equal to 1 by the conditions laid out in Table I. Thus we get from (23):

$$\text{Var}(X) = N^2 \left\{ \frac{1}{k} \sigma_1^2 \left[\frac{p_i}{\pi_i} \sum_2 p_{ij} \sum_3 p_{ijh} \xi_{ijh} \right] \right.$$

$$\left. + \frac{1}{k} \sum_1 \pi_i \sum_2 \frac{\nu_{ij} - n_{ij}}{\nu_{ij} - 1} \cdot \frac{1}{n_{ij}} \sigma_3^2 \left[\frac{p_i}{\pi_i} \frac{p_{ij}}{\pi_{ij}} \frac{p_{ijh}}{\pi_{ijh}} \xi_{ijh} \right] \right\}.$$

Note that generally if the selection in the first stage is made with replacement the estimated variance error, $\exists$ Var (X), by formula (24) has only one term. This recalls the application of the computation technique from an analysis of variance, where the estimated variance error (only if selection is done without replacement) can be calculated from the mean square between primary units. Also it is interesting to note, that in all stages at which selection with replacement is used, the estimates of the parameters of the universe are limited to two terms at most.

If we let the chance of reselection of the same unit in any stage drop to zero as soon as such a unit is selected for the first time, let us call it *complete non-replacement.* In this case formulæ in Section 3 do not apply any more.

Although the estimator (20) is then biassed, we may calculate the estimate according to it. To get an approximation in the other formulæ we have to put $\epsilon = \kappa/K$ for non-replacement in the first stage, and analogously

$$\epsilon_i = \frac{\lambda_i}{L_i}, \quad \epsilon_{ij} = \frac{\nu_{ij}}{N_{ij}}$$

in the second and third stage respectively. So we get approximatively that the estimator of the universe total

$$X = \frac{N}{k} \mathop{S}_1 \frac{p_i}{\pi_i l_i} \mathop{S}_2 \frac{p_{ij}}{\pi_{ij} n_{ij}} \mathop{S}_3 \frac{p_{ijh} x_{ijh}}{\pi_{ijh}}$$

provides a biassed estimate (with unknown bias), and has an approximative variance error:

$$\text{Var}(X) = N^2 \left\{ \frac{1}{k} \frac{K-k}{K-1} \sigma_1^2 \left[\frac{p_i}{\pi_i} \sum_2 p_{ij} \sum_3 p_{ijh} \xi_{ijh} \right] \right.$$

$$+ \frac{1}{k} \sum_1 \pi_i \frac{L_i - l_i}{L_i - 1} \cdot \frac{1}{l_i} \sigma_2^2 \left[\frac{p_i}{\pi_i} \frac{p_{ij}}{\pi_{ij}} \sum_3 p_{ijh} \xi_{ijh} \right]$$

$$+ \frac{1}{k} \sum_1 \pi_i \cdot \frac{1}{l_i} \sum_2 \pi_{ij} \frac{N_{ij} - n_{ij}}{N_{ij} - 1}$$

$$\times \frac{1}{n_{ij}} \sigma_3^2 \left[\frac{p_i}{\pi_i} \cdot \frac{p_{ij}}{\pi_{ij}} \cdot \frac{p_{ijh}}{\pi_{ijh}} \xi_{ijh} \right] \Big\}.$$

and an approximative (biassed) estimate of the variance error:

$$\mathcal{E}\operatorname{Var}(X) = N^2 \left\{ \frac{1}{k} \left(1 - \frac{k}{K}\right) s_1^2 \left[\frac{p_i}{\pi_i l_i} \underset{2}{S} \frac{p_{ij}}{\pi_{ij} n_{ij}} \underset{3}{S} \frac{p_{ijh} x_{ijh}}{\pi_{ijh}} \right] \right.$$

$$+ \frac{1}{kK} \underset{1}{S} \frac{1}{l_i} \left(1 - \frac{l_i}{L_i}\right) s_2^2 \left[\frac{p_i}{\pi_i} \cdot \frac{p_{ij}}{\pi_{ij} n_{ij}} \underset{3}{S} \frac{p_{ijh} x_{ijh}}{\pi_{ijh}} \right]$$

$$+ \frac{1}{kK} \underset{1}{S} \frac{1}{l_i L_i} \underset{2}{S} \frac{1}{n_{ij}} \left(1 - \frac{n_{ij}}{N_{ij}}\right)$$

$$\left. \times s_3^2 \left[\frac{p_i}{\pi_i} \cdot \frac{p_{ij}}{\pi_{ij}} \cdot \frac{p_{ijh}}{\pi_{ijh}} x_{ijh} \right] \right\} \tag{56}$$

These formulæ simplify significantly with selections using probabilities proportionate to size (size being the number of tertiary unit in primary and secondary sampling units). The approximation (56) for two-stage sampling with a selection proportionate to size was given in a similar form by Gray and Cortlett.[7] For purposes of comparison in our formula (56) must be put $\pi_i = p_i, \pi_{ij} = p_{ij}, \pi_{ijh} = p_{ijh}, N_{ij} = n_{ij}$ $\sum_3$ and $\underset{3}{S}$ are reduced to one term, $l_i = l$. It appears that in the formula given by Gray, the factor $L_i/\bar{L}_i$ has no justification, although it does not much affect the order of size of the estimated variance error.

8. Applications in the Case of Linear Biassed Estimates

Suppose that we have selected a three-stage sample with varying probabilities of selection π_i, π_{ij}, π_{ijh}. We estimate the universe total (1) by:

$$X = \frac{N}{k} \underset{1}{S} \frac{1}{l_i} \underset{2}{S} \frac{1}{n_{ij}} \underset{3}{S} x_{ijh} \tag{57}$$

i.e., by an estimator which gives an unbiassed estimate only if the probabilities of selection in every stage are proportionate to the size of units (defined by the number of tertiary units in them). In our case, this estimator yields a biassed estimate. What is its variance error and the estimate of it?

[7] See Ref. [2], page 155.

Comparing (57) with (30), we get that

$$a = \frac{N}{k}, \quad b_i = \frac{1}{l_i}, \quad c_{ij} = \frac{1}{n_{ij}}, \; d_{ijh} = 1.$$

Thus from (31) we get that the estimator above estimates

$$N \sum_1 \pi_i \sum_2 \pi_{ij} \sum_3 \pi_{ijh} \xi_{ijh},$$

which is different from (1).

From (36) we get:

$$\begin{aligned} \text{Var}(X) = N^2 \Bigg\{ & \frac{1}{k} \, \frac{\kappa - \epsilon k}{\kappa - \epsilon} \, \sigma_1^2 \left[\sum_2 \pi_{ij} \sum_3 \pi_{ijh} \xi_{ijh} \right] \\ & + \frac{1}{k} \sum_1 \pi_i \, \frac{\lambda_i - \epsilon_i l_i}{\lambda_i - \epsilon_i} \cdot \frac{1}{l_i} \, \sigma_2^2 \left[\sum_3 \pi_{ijh} \xi_{ijh} \right] \\ & + \frac{1}{k} \sum_1 \pi_i \frac{1}{l_i} \sum_2 \pi_{ij} \, \frac{\nu_{ij} - \epsilon_{ij} n_{ij}}{\nu_{ij} - \epsilon_{ij}} \cdot \frac{1}{n_{ij}} \, \sigma_3^2 \, [{}_i \xi_{jh}] \Bigg\} \end{aligned} \tag{58}$$

and from (39):

$$\begin{aligned} \mathcal{E}\, \text{Var}(X) = N^2 \Bigg\{ & \frac{1}{k} \left(1 - \frac{\epsilon k}{\kappa}\right) s_1^2 \left[\frac{1}{l_i} \underset{2}{S} \frac{1}{n_{ij}} \underset{3}{S} \, x_{ijh} \right] \\ & + \frac{\epsilon}{k\kappa} \underset{1}{S} \frac{1}{l_i} \left(1 - \frac{\epsilon_i l_i}{\lambda_i}\right) s_2^2 \left[\frac{1}{n_{ij}} \underset{3}{S} \, x_{ijh} \right] \\ & + \frac{\epsilon}{k\kappa} \underset{1}{S} \frac{\epsilon_i}{l_i \lambda_i} \underset{2}{S} \frac{1}{n_{ij}} \left(1 - \frac{\epsilon_{ij} n_{ij}}{\nu_{ij}}\right) s_3^2 \, [x_{ijh}] \Bigg\}. \end{aligned} \tag{59}$$

Although the variance error (58) is different than (55) where the use of the estimation formula (57) gives an unbiassed estimate, the estimated variance error, (59), has the same expression as in Chapter 5 provided that the same of chances κ, λ_i, ν_{ij} are kept equal. This is a consequence of the fact that in formula (39) do not appear at all the probabilities π_i, π_{ij}, π_{ijh} actually used in the selection of the sample.

Further, we get, *e.g.*, from the first formula of (40)

$$\underset{123}{\mathcal{E}} \, \sigma_1^2 \left[\sum_2 \pi_{ij} \sum_3 \pi_{ijh} \xi_{ijh} \right]$$

$$= \frac{\kappa - \epsilon}{\kappa} \left\{ s_1^2 \left[\frac{1}{l_i} \underset{2}{S} \frac{1}{n_{ij}} \underset{3}{S} \, x_{ijh} \right] \right.$$

$$-\frac{1}{k}\underset{1}{S}\frac{1}{l_i}\left(1-\frac{\epsilon_i l_i}{\lambda_i}\right)s_2^2\left[\frac{1}{n_{ij}}\underset{3}{S}\;x_{ijh}\right]$$

$$-\frac{1}{k}\underset{1}{S}\frac{\epsilon_i}{l_i\lambda_i}\underset{2}{S}\frac{1}{n_{ij}}\left(1-\frac{\epsilon_{ij}n_{ij}}{\nu_{ij}}\right)s_3^2\;[x_{ijh}]\Big\}$$

Formulæ (58) and (59) were developed by Sukhatme and Panse[8] for the case of selections with equal probabilities in all stages, but using the estimator (57). The results are same if we put

$$\kappa = K,\quad \lambda_i = L_i,\quad \nu_{ij} = N_{ij},\quad \pi_i = \frac{1}{K},$$

$$\pi_{ij} = \frac{1}{L_i},\quad \pi_{ijh} = \frac{1}{N_{ij}},\quad l_i = l,\quad n_{ij} = n.$$

We conclude: Whatever the probabilities used in the selection of the sample, we can always use any estimator, which would give for certain probabilities an unbiassed estimate. Although we get in this way a biassed estimate, *the estimation* of the variance error from the sample does not depend on the probabilities actually used in selection. Provided that the differences between the probabilities actually used and those for which the estimator provides an unbiassed estimate are reasonably small, we can expect that the bias in estimation is small too. It is, however, not excluded that in certain physical situations the bias might be small, although the probabilities actually used are widely different from those for which the estimator provides an unbiassed estimate.[9]

Conclusion

The generalization of the result contained in this paper to a bigger number of sampling stages is obvious. All these developed formulæ can be used with a given sampling design after selecting a sample when we wish, on the basis of actual sample data, to use alternative linear estimators, providing the unbiassed and other different biassed estimates. The case is also covered where, *from data on an* actual sample, using a certain estimator, we wish to estimate the variance error for a larger or smaller sample. But here it was essential to assume that this larger or smaller sample is selected by the same type of selection (with or

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[8] See Ref. [4], pp. 137–38 specially formulæ (18) and (28).

[9] See Ref. [4]. There is shown on one example (mean yields of wheat in Punjab, 1943–44) that the estimator (57) has an almost negligible bias although the selection had been done with equal probabilities in each stage.

without replacement in each stage) and with the same probabilities in each stage as the initial sample. The case where this restriction is dropped will be studied in another paper.

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